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Absence of the absolutely continuous spectrum for Stark–Bloch operators with strongly singular periodic potentials

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Abstract. We prove the absence of the absolutely continuous spectrum for the operator $-\frac{d^2}{dx^2} + \sum_{j\in\mathbb{Z}} \alpha\delta'(x-j) + fx$, $f > 0$ and $\alpha \neq 0$, by means of the crystal momentum representation and the Howland's criterion for Floquet-type operators.

1. Introduction

The Stark–Bloch problem with singular crystal periodic potential was first studied from the qualitative point of view by Berezhkowskii and Ovchinnikov [BO] in the case of a delta-type potential. Subsequently, from the numerical viewpoint, strong evidence for the existence of bound states was given by Ferrari, Grecchi and Zironi at least for small strength of the electric field [FGZ]. At present the only rigorous result on this class of problems is due to Delyon, Simon and Souillard as regards a random δ-type model [DSS], where in particular a transition from point spectrum to continuous spectrum is proved as the electric field strength increases. On the other hand, the rigorous solution for the Stark problem with periodically arranged δ interactions is still lacking.

In recent years attention has been paid to the Stark–Bloch problem with even more singular crystal potentials. In particular Avron, Exner and Last [AEL] considered the case of periodically arranged δ' interactions (with not necessarily identical strengths) and they proved the absence of absolutely continuous spectrum for these models. Their technique consists in regarding the resolvents as performed by trace-class perturbations of a pure point model: then classical results of functional analysis imply the absence of absolutely continuous spectrum.

The aim of our work is to prove the absence of absolutely continuous spectrum by different techniques, in the case in which the crystal potential given by the δ' is exactly periodic. Although this result is contained in the more general results of [AEL], we still think that this procedure is interesting by its relative simplicity and because it can suggest different approaches to the more complex problems of localization in the δ' Stark–Bloch model and of spectral analysis of the δ Stark–Bloch model.

Our proof essentially consists in regarding the original Stark–Bloch operator as a Floquet-type operator via the crystal momentum representation and in verifying that it

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satisfies the hypotheses of Howland's criterion. In general, Floquet operators are formally defined as:

$$K = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + H + V(t)$$

where H is a pure point operator defined on a Hilbert space $\mathcal{H}$ with the simple eigenvalues $\lambda_1 < \lambda_2 < \cdots$, and $V(t)$ is an operator which periodically depends on t. Let us now recall Howland's criterion for bounded periodic perturbation of Floquet operators [H, theorem 1].

Theorem 1. Let $\lambda_{n+1} - \lambda_n \geqslant cn^\gamma$ for some $\gamma > 0$ and $c > 0$ and let V be bounded and strongly C^r in t for $r \geqslant 1 + [\gamma^{-1}]$. Then the operator K defined on $L^2(0, 2\pi) \otimes \mathcal{H}$ has empty absolutely continuous spectrum.

An important role in our proof is played by the fact that the widths of the gaps of the δ' Bloch model are asymptotically linear with respect to the band index, while the widths of the bands are asymptotically constant. With regard to this remark let us stress two facts. First, the same asymptotic behaviour for the gaps and for the bands that occurs when a periodic smooth potential is superimposed on the δ' one still holds by a simple perturbation argument: therefore the same result, namely the absence of the absolutely continuous spectrum is expected (although we do not perform a rigorous proof of that). Besides, in the δ Bloch model the asymptotic behaviour of gaps and bands is the opposite one, so that the associated Stark problem should require a more detailed analysis.

2. The δ' Bloch model

The selfadjoint operator [AGHH]

$$H_B = -\Delta + \sum_{j\in\mathbb{Z}} \alpha\delta'(x - j) \qquad \alpha \neq 0$$

can be defined as the Laplacian on the domain which consists of functions of class $H^{2,2}(\mathbb{R} - \mathbb{Z})$ with the boundary conditions

$$\begin{cases} \varphi'(j+) = \varphi'(j-) = \varphi'(j) \\ \varphi(j+) - \varphi(j-) = \alpha\varphi'(j) \end{cases} \qquad j \in \mathbb{Z} \tag{1}$$

where $'$ denotes the derivative with respect to x.

For the sake of simplicity we can suppose $\alpha = 1$ without loss of generality (except in the particular case $\alpha = -1$, which, however, can be treated analogously). The spectrum of this operator consists of bands

$$\sigma(H_B) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$$

where

$$\beta_n - \alpha_n = 8 - \tfrac{8}{n} + \mathrm{O}(n^{-2}) \tag{2}$$

and

$$\alpha_{n+1} - \beta_n = 2n\pi^2 - 8 + \pi^2 + \tfrac{8}{n} + \mathrm{O}(n^{-2})$$

as n goes to infinity.

The band functions $E_n(k)$ and the Bloch functions $\varphi_n(x, k)$, $k \in [-\pi, \pi]$ periodic in k with period 2π, are the solutions of the eigenvalue equation

$$[-\Delta - E_n(k)]\varphi_n(x, k) = 0 \qquad \varphi_n(x, k) \in L^2([-\tfrac{1}{2}, \tfrac{1}{2}], \mathrm{d}x)$$

with the conditions

$$\begin{aligned} &\varphi_n(-\tfrac{1}{2}, k) = \mathrm{e}^{\mathrm{i}k}\varphi_n(\tfrac{1}{2}, k) \\ &\varphi_n'(-\tfrac{1}{2}, k) = \mathrm{e}^{\mathrm{i}k}\varphi_n'(\tfrac{1}{2}, k) \\ &\varphi_n'(0-, k) = \varphi_n'(0+, k) = \varphi_n'(0, k) \\ &\varphi_n(0+, k) - \varphi_n(0-, k) = \varphi_n'(0, k). \end{aligned} \tag{3}$$

We have that the band functions are the solutions of

$$\cos k = \cos K_n(k) - \frac{K_n(k)}{2}\sin K_n(k) \qquad E_n(k) = K_n^2(k) \tag{4}$$

where one easily finds

$$K_n(k) = n\pi + \mathrm{O}(n^{-1}) \qquad \text{as } n \to \infty. \tag{5}$$

Since H_B is a selfadjoint operator one has that, for fixed k, the Bloch functions φ_n are orthogonal on $[-\frac{1}{2}, \frac{1}{2}]$. On the intervals $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ the Bloch function $\varphi_n(x, k)$ will be a linear combination of the exponentials $\mathrm{e}^{\pm \mathrm{i}K_n(k)x}$:

$$\varphi_n(x, k) = \begin{cases} c_{n+}^- \mathrm{e}^{\mathrm{i}K_n(k)x} + c_{n-}^- \mathrm{e}^{-\mathrm{i}K_n(k)x} & -\frac{1}{2} < x < 0 \\ c_{n+}^+ \mathrm{e}^{\mathrm{i}K_n(k)x} + c_{n-}^+ \mathrm{e}^{-\mathrm{i}K_n(k)x} & 0 < x < \frac{1}{2}. \end{cases} \tag{6}$$

By imposing the conditions (3) one finds

$$\varphi_n(x, k) = C_n(k) \begin{cases} \mathrm{e}^{\mathrm{i}K_n(k)x} - \mathrm{e}^{\mathrm{i}(k - K_n(k))} w_n(k) \mathrm{e}^{-\mathrm{i}K_n(k)x} & -\frac{1}{2} < x < 0 \\ \mathrm{e}^{\mathrm{i}K_n(k)x}\mathrm{e}^{-\mathrm{i}(k + K_n(k))} - w_n(k)\mathrm{e}^{-\mathrm{i}K_n(k)x} & 0 < x < \frac{1}{2} \end{cases} \tag{7}$$

where

$$w_n(k) = \frac{1 - \mathrm{e}^{-\mathrm{i}(k + K_n(k))}}{1 - \mathrm{e}^{\mathrm{i}(k - K_n(k))}}$$

and $C_n(k)$ is the normalization constant. By some calculations and using equation (4) we have that

$$|w_n(k)|^2 = 1 + \frac{\sin k}{\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k) - \sin k}. \tag{8}$$

and

$$C_n(k) = \{2 + d_n(k)\}^{-1/2} \tag{9}$$

where

$$d_n(k) = \frac{\sin k + \frac{1}{2}\sin K_n(k)}{\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k) - \sin k}. \tag{10}$$

By equation (5) it follows that $C_n(k) = \frac{1}{\sqrt{2}} + \mathrm{O}(n^{-1})$ as $n \to \infty$.

We have the following results.

Theorem 2. The Bloch functions $\varphi_n(x,k)$ are a 'complete orthonormal' set in $L^2(\mathbb{R}, \mathrm{d}x)$ in the following sense: given $\varphi \in H_0^{2,2}(\mathbb{R}-\mathbb{Z})$ (where $H_0^{2,2}(\mathbb{R}-\mathbb{Z})$ denotes the space of functions of class $H^{2,2}(\mathbb{R}-\mathbb{Z})$ with compact support in $\mathbb{R}$) and satisfying the conditions (1) one has

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} a_n(k)\varphi_n(x,k)\,\mathrm{d}k$$

where the $a_n(k)$'s ($a_n(k) \in \mathcal{H}_n$, $\mathcal{H}_n$ being the space of periodic functions $L^2([-\pi,\pi], \mathrm{d}k)$ of period 2π) are defined by

$$a_n(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x)\bar{\varphi}_n(x,k)\,\mathrm{d}x.$$

This transformation is an isometry and therefore it admits a continuous extension from the whole $L^2(\mathbb{R})$ onto $\oplus_{n=1}^{\infty}\mathcal{H}_n$.

Proof. The statement can be proved by an adaptation of [RS, theorem XIII.98], taking into account the particular domain of H_B. In fact we remark that the operator $-\Delta$ with the boundary conditions (3) has compact resolvent for each k and that the band functions $E_n(k)$ are non-degenerate (at least in the case $\alpha \neq -1$) [AGHH, theorem III.3.4]. The remaining part of the proof coincides with that of the above-cited theorem. □

3. Absence of the absolutely continuous spectrum for the Stark–Bloch operator

Now let us consider the selfadjoint Stark–Bloch operator

$$H_f = H_B + fx \qquad f > 0$$

on the domain

$$\mathcal{D}(H_f) = \{\varphi \in L^2 : \varphi' \in AC(\mathbb{R}-\mathbb{Z}),\ \varphi \text{ satisfies (1) and, besides, } -\varphi'' + fx\varphi \in L^2(\mathbb{R}-\mathbb{Z})\}.$$

Our aim is to study the spectrum of H_f. Starting from theorem 2 let us consider the crystal momentum representation of the operator H_f, when applied to a function $\varphi \in \mathcal{D}(H_f)$. It takes the form (see Blount [B] for a review)

$$H_f\varphi \longrightarrow \left[\mathrm{i}f\frac{\mathrm{d}}{\mathrm{d}k} + \mathbf{E}(k) + f\mathbf{X}(k)\right]a \qquad a = (a_1,\ldots,a_n,\ldots) \in \oplus_{n=1}^{\infty}\mathcal{H}_n \tag{11}$$

where $\mathbf{E}(k) = \mathrm{diag}(E_n(k))$ is the diagonal matrix defined by the band functions while $\mathbf{X}(k) = (X_{m,n}(k))_{m,n}$ is the inter-band coupling matrix, such that

$$X_{n,m}(k) = \mathrm{i}\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{u}_n(x,k)\frac{\partial u_m(x,k)}{\partial k}\mathrm{d}x. \tag{12}$$

Here $u_n(x,k)$ are the functions, periodic in x with period 1, defined by $\varphi(x,k) = \mathrm{e}^{\mathrm{i}kx}u_n(x,k)$.

Now let us regard the operator (11) the following way:

$$\mathrm{i}f\frac{\mathrm{d}}{\mathrm{d}k} + \mathbf{D} + \mathbf{V}(k)$$

where $\mathbf{D}$ is the diagonal operator whose elements are the numbers $\lambda_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_n(k)\,\mathrm{d}k$ and where $\mathbf{V} = f\mathbf{X} + \mathbf{E} - \mathbf{D}$. Let us first state the following lemma.

Lemma 3. **V** is strongly C^2 in k.

Proof. First we prove that **V** is uniformly bounded in k. On the basis of (4), (5), (7) and (8), the stationary phase theorem gives

$$X_{n,m}(k) = O([K_n(k) - K_m(k)]^{-1}) \leqslant \frac{C_1}{n-m} \qquad \forall n, m \tag{13}$$

uniformly in k for some $C_1 > 0$.

Therefore $\mathbf{X}(k)$ is uniformly norm bounded in k by the estimate

$$\|\mathbf{X}(k)\|^2 = \max_n \sum_{n=1}^{\infty} |X_{n,m}(k)|^2 \leqslant 2C_1^2 \sum_{n=1}^{\infty} \frac{1}{m^2} < \infty.$$

Besides, by virtue of (2), the difference

$$E_n(k) - \lambda_n \leqslant \beta_n - \alpha_n \leqslant C_2 \tag{14}$$

is uniformly bounded for each n by some $C_2 > 0$. Therefore the operator $\mathbf{E}(k) - \mathbf{D}$ is uniformly bounded because

$$\|\mathbf{E} - \mathbf{D}\| = \max_n |E_n(k) - \lambda_n| \leqslant C_2$$

uniformly in k, and the same holds for $\mathbf{V}(k)$. Moreover it turns out that $\mathbf{E} - \mathbf{D}$ is strongly C^2 in k. Indeed, we recall that $K_n(k)$ is determined by the implicit relation

$$\mathcal{F}(k, K) = \cos k - \cos K + \tfrac{K}{2} \sin K = 0$$

which is obtained from $K_0 = n\pi$ and $k_0 = 0$ (if n is even) or $k_0 = \pi$ (if n is odd). By the implicit function theorem it follows that

$$K_n'(k) = -\frac{\frac{\partial \mathcal{F}}{\partial k}}{\frac{\partial \mathcal{F}}{\partial K}} = \frac{\sin k}{\frac{3}{2}\sin K_n(k) + \frac{1}{2} K_n(k) \cos K_n(k)}$$

where, from (5)

$$\tfrac{3}{2}\sin K_n(k) + \tfrac{1}{2}K_n(k)\cos K_n(k) = \tfrac{1}{2}(-1)^n n\pi + O(n^{-1})$$

so that

$$K_n'(k) = \frac{2\sin k}{n\pi}(1 + O(n^{-1})).$$

Besides, by means of (4) and (5) we have

$$\begin{aligned} K_n''(k) &= \frac{\cos k}{\frac{3}{2}\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k)} - K_n'(k)\frac{\sin k[2\cos K_n(k) - \frac{1}{2}K_n(k)\sin K_n(k)]}{(\frac{3}{2}\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k))^2} \\ &= \frac{\cos k}{\frac{3}{2}\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k)} - K_n'(k)\frac{\sin k[\cos K_n(k) - \cos k]}{(\frac{3}{2}\sin K_n(k) + \frac{1}{2}K_n(k)\cos K_n(k))^2} \\ &= 2(-1)^n \cos k \frac{1}{n\pi}(1 + O(n^{-1})). \end{aligned}$$

Therefore

$$E_n'(k) = 2K_n(k)K_n'(k) = 4\sin k + \mathrm{O}(n^{-1})$$

and

$$E_n''(k) = 2K_n(k)K_n''(k) + 2K_n'(k) = 4\cos k(-1)^n + \mathrm{O}(n^{-1}).$$

So there exists $C_3 > 0$ such that

$$|E_n'(k)| \leqslant C_3 \qquad |E_n''(k)| \leqslant C_3 \qquad \forall k \in [-\pi, \pi],\ \forall n.$$

Hence it follows that $E_n(k) - \lambda_n$ is bounded, as well as its first and second derivatives, uniformly with respect to n. In analogous manner one can remark that the estimate (13) still holds for the first and second derivatives of $X_{n,m}(k)$ (eventually changing the constant, which is in any case independent of k and of the indices n and m) in virtue of the stationary phase theorem. Indeed it is sufficient to verify that the first and second derivatives of $d_n(k)$ and $w_n(k)$ are uniformly bounded in k and in n: this in turn can be simply verified from (10) and (8) by virtue of (5). Thus the lemma is proved. □

Therefore we have the following theorem.

Theorem 4. $\sigma_{\mathrm{ac}}(H_f) = \emptyset$ for any $f > 0$.

Proof. Since $\lambda_{n+1} - \lambda_n = 2n\pi^2(1 + \mathrm{O}(n^{-1}))$, then theorem 1 applies, to conclude that H_f has empty absolutely continuous spectrum, because $\mathbf{V}(k)$ is bounded and strongly C^2 in k. □

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